

# Discrete Toda field equations

Ismagil Habibullin

e-mail: ihabib@imat.rb.ru

## Abstract

Discrete analogs of the finite and affine Toda field equations are found corresponding to the Lie algebras of series  $C_N$  and  $\tilde{C}_N$ .

## 1 Introduction

Consider the Toda chain with three discrete independent variables  $u, v, k$  (see [1], [2]):

$$e^{f_{uv}-f_u-f_v+f} = \frac{1 + he^{f_v^1-f_u}}{1 + he^{f_v-f_u^{-1}}}. \quad (1)$$

Here  $f = f(u, v, k)$  is an unknown function, and the following notations are accepted. The upper index is used to indicate shifts with respect to the third variable  $k$  so that  $f^1 = f(u, v, k+1)$  and  $f^{-1} = f(u, v, k-1)$ . The lower index shows shifts of the first and second variables  $f_u = f(u+h, v, k)$ ,  $f_{-u} = f(u-h, v, k)$ ,  $f_v = f(u, v+h, k)$ ,  $f_{-v} = f(u, v-h, k)$ ,  $h$  is the parameter of the grid such that for small values of  $h$  one gets  $f_u - f \simeq \sqrt{h}D_x f$ , and  $f_v - f \simeq \sqrt{h}D_y f$ , and  $f_{uv} - f_u - f_v + f \simeq hD_x D_y f$ , where  $D_x$  and  $D_y$  differential operator with respect to  $x$  and  $y$ . Evidently the continuum limit  $h \rightarrow 0$  of the discrete Toda chain (DTC) (1) gives the usual two-dimensional Toda chain [3]

$$D_x D_y f = e^{f^1-f} - e^{f-f^{-1}}. \quad (2)$$

Chain (DTC) is closely connected with the well known discrete bilinear Hirota-Miwa equation

$$tt_{uv} - t_u t_v = t_v^1 t_u^{-1}. \quad (3)$$

Here the unknown  $t = t(u, v, k)$  depends also upon three independent discrete variables. Miura type transformation  $t^1 = e^f t$  converts the equation (3) into the equation (1).

The chain (1) admits evidently periodical closure constraint  $f(u, v, k) = f(u, v, k+N)$  which reduces it to a finite field equation with dynamical variables  $f(u, v, 1), f(u, v, 2), \dots, f(u, v, N)$ . The other closure is defined by the degeneration points of the chain. For example, the chain (1) truncated by the conditions  $e^{-f(u,v,0)} = 0$  and  $e^{f(u,v,N+1)} = 0$  is an integrable finite system of discrete hyperbolic type equations. But as a rule finite field reductions of chains are not exhausted by degenerate and periodic ones. For instance, the chain (2) admits a large class of reductions connected with the simple and affine Lie algebras [4]. Aforementioned degenerate and periodic reductions correspond to the Lie algebras of series  $A_n$  and  $\tilde{A}_n$ , respectively. The problem of looking for discrete analogs of finite reductions of (2) corresponding to the other Lie algebras of the finite grows is still open (see, for example, [5], and also the surveys [7] and

[8]). In [5] the examples of truncations of (1) connected with special classes of solutions have been discussed. We suggest below discrete analogs of the Toda chain of the series  $C_N, \tilde{C}_N$ .

To shorten the formulae introduce notations  $z(u, v, k) = he^{f_v - f_u^{-1}} + 1$  and  $\Delta f = f_{uv} - f_u - f_v + f$ , then the chain (1) gets the form  $z^1 = e^{\Delta f} z$ . Without loosing the generality one can put  $h = 1$ . Really, the parameter  $h$  can be made equal to one by the following shift  $f(k) \rightarrow f(k) + k \ln h$ .

The main result of the paper is given in the following two statements.

**Proposition 1.** The discrete Toda field equation

$$\begin{aligned} e^{\Delta f^{-1}} &= \frac{z}{z_{-u,v}^1}, \\ e^{\Delta f^j} &= \frac{z^{j+1}}{z^j}, \quad 0 \leq j \leq N-1, \\ e^{\Delta f^N} &= \frac{1}{z^N} \end{aligned} \tag{4}$$

admits the Lax pair (see (54)-(55) below). In the continuum limit  $h \rightarrow 0$  the chain (4) turns into the Toda equation of the series  $C_N$ :

$$\begin{aligned} u(-1) &= -u(0), \\ u_{xy} &= e^{u(k+1)-u(k)} - e^{u(k)-u(k-1)}, \\ e^{u(N+1)} &= 0. \end{aligned} \tag{5}$$

**Proposition 2.** The discrete Toda field equation

$$\begin{aligned} e^{\Delta f^{-1}} &= \frac{z}{z_{-u,v}^1}, \\ e^{\Delta f^j} &= \frac{z^{j+1}}{z^j}, \quad 0 \leq j \leq N-1, \\ e^{\Delta f^N} &= \frac{z_{u,-v}^{N-1}}{z^N}, \end{aligned} \tag{6}$$

admits the Lax pair (see (61)-(62), below). In the continuum limit  $h \rightarrow 0$  it turns into the Toda equation of the series  $\tilde{C}_N$ :

$$\begin{aligned} u(-1) &= -u(0), \\ u_{xy} &= e^{u(k+1)-u(k)} - e^{u(k)-u(k-1)}, \\ u(N) &= -u(N-1) \end{aligned} \tag{7}$$

## 2 Involutions of the associated linear systems

The chain (1) admits the Lax pair consisting of two linear discrete equations [1]

$$\psi_u = e^{f_u - f} \psi - \psi^1, \quad \psi_v = \psi + e^{f_v - f^{-1}} \psi^{-1}. \tag{8}$$

Here the indices of the eigenfunction  $\psi = \psi(u, v, k)$  are for the shifts according to the rule, defined above:  $\psi_u = \psi(u+1, v, k)$ ,  $\psi_v = \psi(u, v+1, k)$ ,  $\psi^1 = \psi(u, v, k+1)$ ,  $\psi^{-1} = \psi(u, v, k-1)$  and so on.

Exclude from the system of equations (8) all the shifts of the third variable. As a result one gets a linear discrete hyperbolic equation

$$\psi_{uv} - \psi_u - e^{f_{uv}-f_v}(\psi_v - z\psi) = 0. \quad (9)$$

Below we will need equations dual to (8) and (9). In order to find the dual equations we use the discrete symmetries of the chain (1). Evidently the chain is invariant under involution  $u \rightarrow 1-v$ ,  $v \rightarrow 1-u$ . However the involution changes the Lax pair (8), which takes now the form

$$y_{-u} = y + e^{f_{-u}-f^{-1}}y^{-1}, \quad y_{-v} = e^{f_{-v}-f}y - y^1. \quad (10)$$

The hyperbolic equation (9) turns into the equation

$$y_{uv} - \frac{1}{z}y_v - \frac{e^{f_v-f}}{z}(y_u - y) = 0. \quad (11)$$

The idea to use two (mutually conjugate) Lax pairs when studying the finite Toda chain belongs to the classical work by Darboux [3].

By analogy with the continuous case pose the question when two hyperbolic equations (9) and (11) are related to each other by a multiplicative transform like  $y = a\psi$  (see, also, [6])? Two hyperbolic equations are connected by a multiplicative transform if and only if their Laplace invariants are the same [9]. Remind that the Laplace invariants of the equation

$$a\psi + b\psi_u + c\psi_v + d\psi_{uv} = 0 \quad (12)$$

are expressed as

$$K_1 = \frac{bc_u}{da_u}, \quad K_2 = \frac{b_v c}{da_v}. \quad (13)$$

Denote through  $K_{1\psi}(u, v, k)$ ,  $K_{2\psi}(u, v, k)$  Laplace invariants of the equation (9) and through  $K_{1y}(u, v, k)$ ,  $K_{2y}(u, v, k)$  – Laplace invariants of the equation (11). Compute all these invariants and find

$$K_{1y} = \frac{1}{z^1}, \quad K_{2y} = \frac{1}{z}, \quad K_{1\psi} = \frac{1}{z_u}, \quad K_{2\psi} = \frac{1}{z_v^1}. \quad (14)$$

Therefore, if the field variables satisfy the constraint

$$z(u+1, v, k_0-1) = z(u, v+1, k_0+1) \quad (15)$$

for a fixed value  $k = k_0$ , then the Laplace invariants of these equations will satisfy the following conditions

$$K_{1y}(u+1, v, k_0-1) = K_{1\psi}(u, v, k_0), \quad K_{2y}(u+1, v, k_0-1) = K_{2\psi}(u, v, k_0), \quad (16)$$

and the conditions

$$K_{1\psi}(u, v-1, k_0-1) = K_{1y}(u, v, k_0), \quad K_{2\psi}(u, v-1, k_0-1) = K_{2y}(u, v, k_0). \quad (17)$$

Consequently, there are such functions  $R = R(u, v)$ ,  $S = S(u, v)$  that the following relations take place

$$\psi(u, v-1, k_0-1) = Ry(u, v, k_0) \quad \text{and} \quad y(u+1, v, k_0-1) = S\psi(u, v, k_0) \quad (18)$$

between solutions of the hyperbolic equations (9) and (11). Multipliers  $R$  and  $S$  are found from the following overdetermined system of linear equations

$$R_u = Rz_{-v}e^{f_{u,-v}^{-1}-f_{-v}^{-1}}, \quad R_v = Rz_{-u}e^{f_{-u}^{-1}-f_{-u,v}^{-1}}, \quad (19)$$

$$S_u = S\frac{e^{f_{-u}^{-1}-f_u^{-1}}}{z_{-u,-v}^{-1}}, \quad S_v = S\frac{e^{f_v^{-1}-f^{-1}}}{z^{-1}}, \quad (20)$$

consistency of which is guaranteed by the condition (15).

### 3 Lax pair of the semi-infinite chain

Equations (18) can be referred to as boundary conditions cutting off the linear equations (8), (10) reducing them into the half-line  $k \geq k_0$  (or the half-line  $k \leq k_0$ ). Concentrate on this statement. Put first for the simplicity  $k_0 = 0$ . Substituting (18) into the equations (8) and (10) for  $k = 0$  one gets

$$y_{-u}^0 = y^0 + X_{-u}\psi_{-u}^0, \quad \psi_v^0 = \psi^0 + H_v y_v^0, \quad (21)$$

where  $X = Se^{f-f_u^{-1}}$ ,  $H = Re^{f-f_v^{-1}}$ .

The following lemma gives the connection between the functions  $X$  and  $H$ .

**Lemma 1.** Solutions of the system of the equations (19), (20) can be chosen to satisfy the constraint

$$HX = \frac{z_{-v} - 1}{z_{-v}}. \quad (22)$$

Proof. It follows from the equations (19) and (20) that functions  $X$  and  $H$  satisfy the similar linear difference equations of the first order

$$X_v = X\frac{e^{f_v-f}}{z}, \quad H_v = Hz_{-u}^1e^{f_{-v}^{-1}-f^{-1}}, \quad (23)$$

$$X_u = X\frac{e^{f_u^{-1}-f_{u^2}^{-1}}}{z^1}, \quad X_v = X\frac{e^{f_v-f}}{z}, \quad (24)$$

where  $f_{u^2}^{-1} := f(u+2, v, k-1)$ .

It is not difficult to check that if the condition (22) holds at a point  $(u, v)$  then it also holds at any neighbouring point.

Let us discuss briefly dynamical variables. Evidently shifts of the eigenfunction  $\psi$  in the positive direction:  $\psi_u, \psi_{u^2}, \psi_v, \psi_{v^2}, \dots$  as well as shifts of the eigenfunction  $y$  in the negative direction such as  $y_{-u}, y_{-u^2}, y_{-v}, y_{-v^2}, \dots$  are locally expressed through unshifted (dynamical) variables  $\psi, \psi^{\pm 1}, \psi^{\pm 2}, \dots$  and  $y, y^{\pm 1}, y^{\pm 2}, \dots$ . This is not the case for the shifts on the opposite directions which really should be considered as nonlocal variables. Actually these variables cannot be expressed through a finite number of dynamical ones. For example, to find the variable  $\psi_{-u}$  it is necessary to solve the difference equation (with respect to the argument  $k$ ) of the form

$$\psi_{-u}^1 - e^{f-f_{-u}}\psi_{-u} = \psi.$$

Enlarge the set of dynamical variables. In addition to the set of dynamical variables on the half-line  $k \geq 0$ , consisting of the functions  $\{\psi^0, y^0, \psi^1, y^1, \dots\}$ , introduce two more variables  $Y$  and  $\Psi$  by setting  $Y = y_v^0$ ,  $\Psi = \psi_{-u}^0$ .

**Lemma 2.** The shifts  $Y_{-u}$ ,  $Y_{-v}$ ,  $Y_u$  of the variable  $Y$  as well as the shifts  $\Psi_u$ ,  $\Psi_v$ ,  $\Psi_{-v}$  of the variable  $\Psi$  are linearly expressed through a finite number of the elements of the enlarged dynamical set  $\{\Psi, Y, \psi^0, y^0, \psi^1, y^1, \dots\}$ .

Proof. Some of the equations required follows directly from the definition:  $Y_{-v} = y^0$ ,  $\Psi_u = \psi^0$ . Let us shift the first equation (21) to the right by one with respect to  $u$  and to  $v$  and rewrite it in the form

$$Y_u = Y - X_v \psi_v^0 = Y - X_v(\psi^0 + H_v Y) = Y(1 - X_v H_v) - X_v \psi^0.$$

Due to the Lemma 1 the quantity inside the parentheses is equal to  $\frac{1}{z}$ , hence

$$Y_u = \frac{1}{z} Y - X_v \psi^0.$$

Shift the expression obtained to the left by one respect to  $u$  and transform it as

$$Y_{-u} = z_{-u} Y + X_{-u,v} z_{-u} \Psi. \quad (25)$$

Shift now the second equation of (21) by one to the left respect  $u$  and  $v$  then after slight simplification one gets

$$\Psi_{-v} = \frac{1}{z_{-u,-v}} \Psi - H_{-u} y^0,$$

and

$$\Psi_v = z_{-u} \Psi + H_{-u,v} z_{-u} Y. \quad (26)$$

Lemma 2 is proved.

The commentary to the lemma. Variables  $\psi^j$  can be shifted upward and to the right on the  $(u, v)$ -plane while the variables  $y^j$  – downward and to the left. The new variables are special – they can be shifted on three directions.

Summarize the computations above. Introduce some notations. Denote through  $P$  and  $Q$  infinite dimensional vectors-columns such as

$$P = \begin{pmatrix} Y \\ y^0 \\ y^1 \\ y^2 \\ \dots \end{pmatrix}, \quad Q = \begin{pmatrix} \Psi \\ \psi^0 \\ \psi^1 \\ \psi^2 \\ \dots \end{pmatrix}, \quad (27)$$

i.e.  $P_0 = Y$ ,  $Q_0 = \Psi$  and  $Q_i = \psi^{i-1}$  for  $i \geq 1$   $P_i = y^{i-1}$ . Introduce six infinite dimensional matrices, four of which are two diagonal

$$A = \begin{pmatrix} z_u & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & e^{f_{-u}^1 - f} & 1 & 0 & \dots \\ 0 & 0 & e^{f_{-u}^2 - f^1} & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & e^{f_{-v} - f} & -1 & 0 & \dots \\ 0 & 0 & e^{f_{-v}^1 - f^1} & -1 & \dots \\ 0 & 0 & 0 & e^{f_{-v}^2 - f^2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (28)$$

$$C = \begin{pmatrix} z_v & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & e^{f_v^1 - f} & 1 & 0 & \cdots \\ 0 & 0 & e^{f_v^2 - f^1} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & e^{f_u - f} & -1 & 0 & \cdots \\ 0 & 0 & e^{f_u^1 - f^1} & -1 & \cdots \\ 0 & 0 & 0 & e^{f_u^2 - f^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (29)$$

and two others are of the form

$$a = \begin{pmatrix} X_{-u,v} z_{-u} & 0 & \cdots \\ X_{-u} & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}, \quad c = \begin{pmatrix} H_{-u,v} z_v & 0 & \cdots \\ H_v & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}. \quad (30)$$

The last two matrices have all entries to be zero except the following ones  $a_{0,0}$ ,  $a_{1,0}$ ,  $c_{0,0}$ ,  $c_{1,0}$ .

Make up a system of difference equations

$$P_{-u} = AP + aQ, \quad P_{-v} = BP, \quad (31)$$

$$Q_v = CQ + cP, \quad Q_u = DQ. \quad (32)$$

**Proposition 3.** The systems (31) and (32) are consistent if and only if their coefficients satisfy the semi-infinite lattice of equations

$$e^{\Delta f^{-1}} = \frac{z}{z_{-u,v}^1}, \quad (33)$$

$$e^{\Delta f^j} = \frac{z^{j+1}}{z^j}, \quad \text{for } 0 \leq j < \infty. \quad (34)$$

To prove the proposition it is enough to check that the conditions  $(P_{-u})_{-v} = (P_{-v})_{-u}$ ,  $(Q_u)_v = (P_v)_u$  hold only together with the constraints (33), (34). It is clear that the system (31), (32) is the matrix form of the following system of the scalar equations

$$\psi_u^j = e^{f_u^j - f^j} \psi^j - \psi^{j+1}, \quad y_{-v}^j = e^{f_{-u}^j - f^j} y^j - y^{j+1}, \quad \text{for } j \geq 0, \quad (35)$$

$$\psi_v^j = \psi^j + e^{f_v^j - f^{j-1}} \psi^{j-1}, \quad y_{-u}^j = y^j + e^{f_{-u}^j - f^{j-1}} y^{j-1}, \quad \text{for } j \geq 1, \quad (36)$$

$$y_{-u}^0 = y^0 + X_{-u} \psi_{-u}^0, \quad \psi_v^0 = \psi^0 + H_v y_v^0, \quad (37)$$

$$Y_{-u} = z_{-u} Y + X_{-u,v} z_{-u} \Psi, \quad \Psi_v = z_{-u} \Psi + H_{-u,v} z_{-u} Y, \quad (38)$$

$$Y_{-v} = y^0, \quad \Psi_u = \psi^0. \quad (39)$$

In other words one has to check validity of the conditions  $(Y_{-u})_{-v} = (Y_{-v})_{-u}$ ,  $(\Psi_u)_v = (\Psi_v)_u$ ,  $(y_{-u}^j)_{-v} = (y_{-v}^j)_{-u}$ ,  $(\psi_u^j)_v = (\psi_v^j)_u$ . Moreover, it is enough to take  $j = 0$ , because for the other  $j$  it is really true. The conditions required are equivalent the following five equations:

$$X_v = X \frac{e^{f_v - f}}{z},$$

$$z^1 e^{-\Delta f} = 1 + (z - 1) \frac{z_{-u}^1}{z_{-v}^1},$$

$$X_v = X \frac{e^{f_v - f}}{z},$$

$$z^1 = z e^{\Delta f},$$

$$HX = \frac{z_{-v} - 1}{z_{-v}}.$$

The second and fourth of them are the consequences of the boundary condition  $z_v^1 = z_u^{-1}$  and the others have already been proved to be consistent under the boundary condition.

In order to find the continuum limit as  $h \rightarrow 0$  rewrite the boundary condition  $z_v^1 = z_u^{-1}$  in a more explicit form

$$f_{u,-v}^{-2} = f^{-1} + f - f_{-u,v}^1. \quad (40)$$

Hence  $f_u = f + hD_x f + O(h^2)$  and  $f_v = f + hD_y f + O(h^2)$  then setting  $h \rightarrow 0$  one gets  $f^{-2} = f^{-1} + f - f^1$ . Then it follows from (2) that  $D_x D_y (f + f^{-1}) = 0$ . Consequently,  $f = -f_{-1} + a(x) + b(y)$ . Remove functions  $a(x)$  and  $b(y)$  by the dilatation  $f = \tilde{f} + a(x)/2 + b(y)/2$ . As a result one gets the boundary condition searched  $f^{-1} = -f$ .

Notice that when the additional constraint  $f(u, v, k) = f(u + v, k)$  is imposed on the chain (1), which reduces it to 1+1 dimensional discrete Toda chain, the boundary condition (40) is reduced to the form  $f(u, -1) = -f(u, 0)$  found earlier by Yu.Suris in [10] (see also [11]).

Consider now the other half of the chain which is located on the left half-line  $k \leq k_0$ . Formulae (18) allow one to exclude from the Lax pair the functions  $y^0$  and  $\psi^0$ , and then rewrite the shifted variables  $y_{-v}^{-1}$  and  $\psi_u^{-1}$  in the form

$$y_{-v}^{-1} = y^{-1} e^{f_{-v}^{-1} - f^{-1}} - \frac{1}{R} \psi_{-v}^{-1}, \quad \psi_u^{-1} = \psi^{-1} e^{f_u^{-1} - f^{-1}} - \frac{1}{S} y_u^{-1}. \quad (41)$$

Introduce additional dynamical variables  $\Psi^{-1} := \psi_{-v}^{-1}$  and  $Y^{-1} := y_u^{-1}$ . Then the previous equation can be transformed as follows

$$y_{-v}^{-1} = y^{-1} e^{f_{-v}^{-1} - f^{-1}} - \frac{1}{R} \Psi^{-1}, \quad \psi_u^{-1} = \psi^{-1} e^{f_u^{-1} - f^{-1}} - \frac{1}{S} Y^{-1}. \quad (42)$$

Below we will use the following analog of Lemma 1.

**Lemma 3.** Solutions of the equations (19) and (20) can be chosen to satisfy the constraint

$$R_{uv} S = \frac{z}{z-1}. \quad (43)$$

To prove the lemma one has to express  $H$  and  $X$  through  $R$  and  $S$  and substitute them into (22).

**Lemma 4..** The shifts  $Y_{-u}^{-1}$ ,  $Y_{-v}^{-1}$ ,  $Y_v^{-1}$ ,  $\Psi_u^{-1}$ ,  $\Psi_v^{-1}$ ,  $\Psi_{-u}^{-1}$  of the variables  $Y^{-1}$  and  $\Psi^{-1}$  are linearly expressed through the finite number of elements of the enlarged dynamical set  $\{\Psi^{-1}, Y^{-1}, \psi^{-1}, y^{-1}, \psi^{-2}, y^{-2}, \dots\}$ :

$$\begin{aligned} Y_{-u}^{-1} &= y^{-1}, \\ Y_v^{-1} &= \frac{1}{z} e^{f_{uv}^{-1} - f_u^{-1}} Y^{-1} + \frac{1}{R_{uv}} e^{f_{uv}^{-1} - f^{-1}} \psi^{-1}, \\ Y_{-v}^{-1} &= z_{-v} e^{f_{u,-v}^{-1} - f_u^{-1}} Y^{-1} - \frac{z_{-v}}{R_u} e^{f_{u,-v}^{-1} - f_{-v}^{-1}} \Psi^{-1}, \\ \Psi_v^{-1} &= \psi^{-1}, \\ \Psi_{-u}^{-1} &= \frac{1}{z_{-u,-v}} e^{f_{-u,-v}^{-1} - f_{-v}^{-1}} \Psi^{-1} + \frac{1}{S_{-u,-v}} e^{f_{-u,-v}^{-1} - f^{-1}} y^{-1}, \\ \Psi_u^{-1} &= z_{-v} e^{f_{u,-v}^{-1} - f_{-v}^{-1}} \Psi^{-1} - \frac{z_{-v}}{S_{-v}} e^{f_{u,-v}^{-1} - f_u^{-1}} Y^{-1}. \end{aligned}$$

Lemma 4 is proved similarly to the lemma 2.

Introduce infinite dimensional vectors-columns  $\tilde{P}$  and  $\tilde{Q}$  as

$$\tilde{P} = \begin{pmatrix} \dots \\ y^{-3} \\ y^{-2} \\ y^{-1} \\ Y^{-1} \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} \dots \\ \psi^{-3} \\ \psi^{-2} \\ \psi^{-1} \\ \Psi^{-1} \end{pmatrix}, \quad (44)$$

and infinite dimensional matrices

$$\tilde{A} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 0 \\ \dots & e^{f_u^{-2}-f^{-3}} & 1 & 0 & 0 \\ \dots & 0 & e^{f_u^{-1}-f^{-2}} & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (45)$$

$$\tilde{B} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & e^{f_v^{-3}-f^{-3}} & -1 & 0 & 0 \\ \dots & 0 & e^{f_v^{-2}-f^{-2}} & -1 & 0 \\ \dots & 0 & 0 & e^{f_v^{-1}-f^{-1}} & 0 \\ \dots & 0 & 0 & 0 & z_{-v}e^{f_{u,-v}^{-1}-f_u^{-1}} \end{pmatrix} \quad (46)$$

$$\tilde{C} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 0 \\ \dots & e^{f_v^{-2}-f^{-3}} & 1 & 0 & 0 \\ \dots & 0 & e^{f_v^{-1}-f^{-2}} & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (47)$$

$$\tilde{D} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & e^{f_u^{-3}-f^{-3}} & -1 & 0 & 0 \\ \dots & 0 & e^{f_u^{-2}-f^{-2}} & -1 & 0 \\ \dots & 0 & 0 & e^{f_u^{-1}-f^{-1}} & 0 \\ \dots & 0 & 0 & 0 & z_{-v}e^{f_{u,-v}^{-1}-f_v^{-1}} \end{pmatrix} \quad (48)$$

and two more matrices each with only two nonzero entries

$$\tilde{b} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & 0 & 0 \\ \dots & 0 & -\frac{1}{R} \\ \dots & 0 & -\frac{z_{-v}}{R_u}e^{f_{u,-v}^{-1}-f_v^{-1}} \end{pmatrix}, \quad \tilde{d} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & 0 & 0 \\ \dots & 0 & -\frac{1}{S} \\ \dots & 0 & -\frac{z_{-v}}{S_{-v}}e^{f_{u,-v}^{-1}-f_u^{-1}} \end{pmatrix}. \quad (49)$$

**Proposition 4.** The following system of equations

$$\tilde{P}_{-u} = \tilde{A}\tilde{P}, \quad \tilde{P}_{-v} = \tilde{B}\tilde{P} + \tilde{b}\tilde{Q}, \quad (50)$$

$$\tilde{Q}_v = \tilde{C}\tilde{Q}, \quad \tilde{Q}_u = \tilde{D}\tilde{Q} + \tilde{d}\tilde{P} \quad (51)$$



is consistent if and only if the function  $f = f(u, v, k)$  solves the semi-infinite lattice

$$e^{\Delta f} = \frac{z_{u,-v}^{-1}}{z}, \quad (52)$$

$$e^{\Delta f^j} = \frac{z^{j+1}}{z^j}, \quad \text{for } -\infty < j \leq -1. \quad (53)$$

## 4 Finite chains

Close the semi-infinite chain (33), (34) by imposing an additional (degenerate) boundary condition  $e^{f^{N+1}} = 0$ . The obtained chain coincides with (4). One can close also the Lax pair by setting  $y^{N+1} = 0$ ,  $\psi^{N+1} = 0$ . The Lax pair found can be represented in the matrix form

$$P_{-u} = AP + aQ, \quad P_{-v} = BP, \quad (54)$$

$$Q_v = CQ + cP, \quad Q_u = DQ. \quad (55)$$

with the following matrix coefficients of dimension  $(N+2) \times (N+2)$  :

$$A = \begin{pmatrix} z_u & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{f_u^1 - f} & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{f_u^{N-1} - f^{N-2}} & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_u^N - f^{N-1}} & 1 \end{pmatrix}, \quad (56)$$

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & e^{f_{-v} - f} & -1 & \cdots & 0 & 0 \\ 0 & 0 & e^{f_{-v}^1 - f^1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{f_{-v}^{N-1} - f^{N-1}} & -1 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_{-v}^N - f^N} \end{pmatrix}, \quad (57)$$

$$C = \begin{pmatrix} z_v & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{f_v^1 - f} & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{f_v^{N-1} - f^{N-2}} & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_v^N - f^{N-1}} & 1 \end{pmatrix}, \quad (58)$$

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & e^{f_u - f} & -1 & \cdots & 0 & 0 \\ 0 & 0 & e^{f_u^1 - f^1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{f_u^{N-1} - f^{N-1}} & -1 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_u^N - f^N} \end{pmatrix}. \quad (59)$$

The matrices  $a$  and  $b$  are also of the dimension  $(N+2) \times (N+2)$ . Their entries are all zero except the first two entries of the first columns:  $a_{00} = X_{-u,v}z_{-u}$ ,  $a_{10} = X_{-u}$ ,  $c_{00} = H_{-u,v}z_v$ ,  $c_{10} = H_v$ .

If one imposes the condition (15) on both ends of the chain, one gets the chain

$$\begin{aligned} e^{\Delta f^{-1}} &= \frac{z}{z_{-u,v}^1}, \\ e^{\Delta f^j} &= \frac{z^{j+1}}{z^j}, \quad \text{for } 0 \leq j \leq N-1, \\ e^{\Delta f^N} &= \frac{z_{u,-v}^{N-1}}{z^N} \end{aligned} \quad (60)$$

(see chain (6) in Introduction). Its Lax pair is given by

$$\hat{P}_{-u} = \hat{A}\hat{P} + \lambda\hat{a}\hat{Q}, \quad \hat{P}_{-v} = \hat{B}\hat{P} + \hat{b}\hat{Q}, \quad (61)$$

$$\hat{Q}_v = \hat{C}\hat{Q} + \lambda^{-1}\hat{c}\hat{P}, \quad \hat{Q}_u = \hat{D}\hat{Q} + \hat{d}\hat{P}, \quad (62)$$

where  $\lambda$  is the spectral parameter and

$$\hat{A} = \begin{pmatrix} z_u & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & e^{f_{-u}^1 - f} & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & e^{f_{-u}^{N-2} - f^{N-3}} & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_{-u}^{N-1} - f^{N-2}} & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (63)$$

$$\hat{B} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{f_{-v} - f} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & e^{f_{-v}^1 - f^1} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & e^{f_{-v}^{N-2} - f^{N-2}} & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_{-v}^{N-1} - f^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & z_{-v}^N e^{f_{u,-v}^{N-1} - f_u^{N-1}} \end{pmatrix}, \quad (64)$$

$$\hat{C} = \begin{pmatrix} z_v & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & e^{f_v^1 - f} & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & e^{f_v^{N-2} - f^{N-3}} & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_v^{N-1} - f^{N-2}} & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (65)$$

$$\hat{D} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{f_u-f} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & e^{f_u^1-f^1} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & e^{f_u^{N-2}-f^{N-2}} & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{f_u^{N-1}-f^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & z_{-v}^N e^{f_{u,-v}^{N-1}-f_{-v}^{N-1}} \end{pmatrix}, \quad (66)$$

$$\hat{a} = \begin{pmatrix} X_{-u,v} z_{-u} & 0 & \cdots & 0 \\ X_{-u} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} H_{-u,v} z_v & 0 & \cdots & 0 \\ H_v & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (67)$$

$$\hat{b} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\frac{1}{R_u^N} \\ 0 & \cdots & 0 & -\frac{z_{-v}^N}{R_u^N} e^{f_{u,-v}^{N-1}-f_{-v}^{N-1}} \end{pmatrix}, \quad \hat{d} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\frac{1}{S_v^N} \\ 0 & \cdots & 0 & -\frac{z_{-v}^N}{S_v^N} e^{f_{u,-v}^{N-1}-f_u^{N-1}} \end{pmatrix}, \quad (68)$$

where  $R^N$  and  $S^N$  are defined through equations

$$R_u^N = R^N z_{-v}^N e^{f_{u,-v}^{N-1}-f_{-v}^{N-1}}, \quad R_v^N = R^N z_{-u}^N e^{f_{-u}^N-f_{-u,v}^N}, \quad (69)$$

$$S_u^N = S^N \frac{e^{f_u^N-f_u^N}}{z_{-u,-v}^{N-1}}, \quad S_v^N = S^N \frac{e^{f_v^{N-1}-f^{N-1}}}{z^{N-1}}, \quad (70)$$

$$R_{uv}^N S^N = \frac{z^N}{z^N - 1}. \quad (71)$$

Explain how the spectral parameter has been introduced. Evidently each of the functions  $H$  and  $X$  is defined by the equations (23), (24) up to the constant multiplier. Due to the constraint (22) imposed above, the only constant remains free, and it is taken as the spectral parameter.

**Acknowledgments.** The work has been supported by grants RFBR#04-01-00190 and RFBR#05-01-97910-r-agidel-a.

## References

- [1] R.Hirota, J.Phys. Soc. Jpn. 50 (1981) 3785.
- [2] D.Levi, L.Pilloni and P.M.Santini, J.Phys. A 14(1981) 1567.
- [3] G. Darboux, Leçons sur la théorie générale des surfaces et les applications géométrique du calcul infinitésimal, T.2. Paris:Goutier-Villars, 1915

- [4] Leznov A.N., Savel'ev M.V., Group methods of integration of nonlinear dynamical systems. M.:Nauka, 1985 (in Russian).
- [5] R.S.Ward, Discrete Toda field equations, Phys. Letts A 199(1995), 45-48.
- [6] I.T.Habibullin, Truncations of Toda chain and the reduction problem, Theoret. and Math. Physics, 143(1): 515-528 (2005); I.T.Habibullin, Multidimensional integrable boundary problems, nlin.SI/0401028.
- [7] A.V.Zabrodin, A survey of Hirota's Difference equations, Theoret. and Math. Physics,(in Russian) 113 (1997) 179-230.
- [8] A.V.Zabrodin, Hirota's equation and the Bethe anzats, Theoret. and Math. Physics,(in Russian) 116 (1998) 54-100.
- [9] I.A.Dynnikov, C.P.Novikov, Russian Math. Survey, 1997, v.32, N5, p.175.
- [10] Yu.B.Suris, Leningrad Math. J. 2(1990) 339.
- [11] Kazakova T G, Finite dimensional discrete systems, integrated in quadratures. *Theoretical and Mathematical Physics* [in Russian] **138**, (2004), 422-436.